CAUCHY'S CONDENSATION TEST

(B.Sc.-II, Paper-III)

Group- B

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(Cauchy's Condensation test)

Statement :→

Let $\{f(1), f(2), \ldots, f(n), \ldots\}$ be a sequence of positive terms such that

$f(1) \geq f(2) \geq f(3) \geq \ldots \geq f(n) \geq \ldots$

for all n. Then the series $\sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge together (where a is any positive integer > 1).

Proof :→

$\because \sum f(n) = f(1) + f(2) + \ldots + f(n) + \ldots$

is a series of positive terms.

$\therefore$ We can group (bracket) the terms of series without changing its nature.

We group the terms of $\sum f(n)$ as follows:

$$\{f(1) + f(2) + \ldots + f(a)\}$$
$$+ \{f(a+1) + f(a+2) + \ldots + f(a^2)\}$$
$$+ \{f(a^2+1) + f(a^2+1) + \ldots + f(a^3)\}$$
$$+ \ldots$$
$$+ \ldots$$

If v_n denotes the sum of the terms of the nth group, then

$$v_n = f(a^{n-1}+1) + f(a^{n-1}+2) + \cdots\cdots + f(a^n).$$

Now, since $f(n)$ is decreasing.

$\therefore$ $f(a^n) \leq$ each term of $v_n \leq f(a^{n-1})$

Also the number of terms in the nth group $= a^n - a^{n-1}$.

$$\therefore \ (a^n - a^{n-1}) f(a^n) \leq v_n \leq (a^n - a^{n-1}) f(a^{n-1}).$$

$$\therefore \ v_n \leq (a-1) a^{n-1} f(a^{n-1}) \quad \text{——— (1) and}$$

$$v_n \geq \frac{(a-1)}{a} a^n f(a^n) \quad \text{——— (2)}$$

From (i), if $\Sigma a^n f(a^n)$ is convergent i.e. if $\Sigma a^{n-1} f(a^{n-1})$ is convergent then Σv_n is convergent by comparision test. ($\because a-1>0$)

From (ii), if $\Sigma a^n f(a^n)$ is divergent then Σv_n is divergent by comparision test.
($\because \frac{a-1}{a} > 0$)

$\because$ The positive term series is either convergent or divergent.

$\therefore$ The two series Σv_n and $\Sigma a^n f(a^n)$ converge or diverge together.

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$\therefore \sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge together.

proved.

Example: $\rightarrow$ Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $p \leq 1$.

Proof: $\rightarrow$ Let $f(n) = \frac{1}{n(\log n)^p}$

Case-I: $\rightarrow$ $(p>0)$

In this case

$$\frac{1}{n(\log n)^p} > \frac{1}{(n+1)\{\log(n+1)\}^p} \; ; \forall \; n \geq 2$$

$\therefore f(n)$ is a decreasing sequence of positive terms.

Let $a > 1$.

Then, $a^n f(a^n) = a^n \cdot \frac{1}{a^n (\log a^n)^p}$

$$= \frac{1}{(\log a^n)^p} = \frac{1}{(n \log a)^p}$$

$$= \frac{1}{n^p} \cdot \frac{1}{(\log a)^p}$$

$\therefore$ By Cauchy's condensation test

$\because \sum \frac{1}{n^p}$ is convergent if $p>1$.

Therefor $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent for $p>1$.

And since $\sum \frac{1}{n^p}$ is divergent for $p \leq 1$.

$\therefore$ The series $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}$ is divergent

for $0 < p \leq 1$.

Case-2 :→ Let $p \leq 0$.

In this case

$$\frac{1}{n(\log n)^p} \geq \frac{1}{n}$$

$\therefore$ By comparision test

The series $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}$ is divergent. #

Cauchy's condensation test

Example ② :→ Discuss the convergence of

$$\sum_{m}^{\infty} \frac{1}{n\log n\ (\log\log n)^{p}}$$ for ~~such~~ suitable m.

Solution:→

The summation must start from n=m for which the terms are defined.

Let $f(n) = \dfrac{1}{n(\log n)\ (\log\log n)^{p}}$

Therefor for any positive integer a>1.

$$a^{n} f(a^{n}) = a^{n} \cdot \frac{1}{a^{n}\ (\log a^{n})\ (\log\log a^{n})^{p}}$$

$$= \frac{1}{(n\log a)\ \{\log(n\log a)\}^{p}}$$

$$= \frac{1}{(n\log a)\ \{\log n + \log\log a\}^{p}}$$

$$= \frac{1}{(n\log a)} \times \frac{1}{(\log n)^{p} \left\{1 + \dfrac{\log\log a}{\log n}\right\}^{p}}$$

$= u_n$ (say)

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Let $v_n = \dfrac{1}{n(\log n)^p}$

Then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{n\cdot(\log n)^p}{n\cdot\log a\cdot(\log n)^p\left\{1+\dfrac{\log\log a}{\log n}\right\}^p}$

$= \dfrac{1}{\log a}$ ($\neq 0$, and finite)

$\therefore$ By comparision test

(i) if $\sum v_n$ is convergent then $\sum u_n$ i.e. $\sum a^n f(a^n)$ is convergent and so $\sum f(n)$ is convergent by cauchy's condensation test.

(ii) if $\sum v_n$ is divergent then $\sum u_n$ i.e, $\sum a^n f(a^n)$ is divergent and so $\sum f(n)$ is divergent by cauchy's condensation test.

But since $\sum v_n = \sum \dfrac{1}{n(\log n)^p}$ is convergent for $p>1$ and divergent for $p\leq 1$.

$\therefore$ $\sum f(n)$ is convergent if $p>1$ and divergent if $p\leq 1$.

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Cauchy's condensation test

Example ① discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$

Solution:→ Let $u_n = \frac{1}{\log n}$

$\because \{u_n\}$ is a decreasing sequence.

i.e. $u_n > u_{n+1} > 0, \ \forall \ n \geq 2.$

$\because$ By cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} a^n u_{a^n}$ converge or diverge together. (where $a > 1$).

Now, $\sum_{n=2}^{\infty} a^n u_{a^n} = \sum_{n=2}^{\infty} a^n \frac{1}{\log a^n}$

$$= \sum_{n=2}^{\infty} \frac{a^n}{n \log a} = \frac{1}{\log a} \sum_{n=2}^{\infty} \frac{a^n}{n}.$$

Let $v_n = \frac{a^n}{n}$

$\therefore \lim_{n\to\infty} (v_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a}{n^{\frac{1}{n}}} = a > 1.$

$\therefore$ By cauchy's root test,

$\sum v_n$ is divergent

$\Rightarrow \sum_{n=2}^{\infty} a^n u_{a^n}$ is divergent $\Rightarrow \sum_{n=2}^{\infty} u_n$ is divergent.

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Example②:→ Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\log n}$.

Solution:→ Let $f(n) = \frac{1}{n\log n}$.

$\because$ $\{n\log n\}$ is an increasing sequence,

$\therefore$ $\{f(n)\}$ is a decreasing sequence.

i.e. $f(n) > f(n+1) > 0$, $\forall\, n \geq 2$.

$\therefore$ By cauchy's condensation test, the series $\sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge together (where $a>1$).

Now, $$\sum_{n=2}^{\infty} a^n f(a^n) = \sum_{n=2}^{\infty} a^n \frac{1}{a^n \cdot \log a^n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n\log a} = \frac{1}{\log a} \sum_{n=2}^{\infty} \frac{1}{n}$$

$\because$ $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent $\Rightarrow$ $\sum_{n=2}^{\infty} a^n f(a^n)$ is divergent

$\Rightarrow$ $\sum_{n=2}^{\infty} f(n)$ is divergent

$\Rightarrow$ $\sum_{n=2}^{\infty} \frac{1}{n\log n}$ is divergent.

Example ⑤:→ Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\log n}}$.

Solution:→ Let $f(n) = \frac{1}{n\sqrt{\log n}}$

$\because$ $\{n\sqrt{\log n}\}$ is an increasing sequence.

$\therefore$ $\{f(n)\}$ is a decreasing sequence.

i.e., $f(n) > f(n+1) > 0, \ \forall \ n \geq 2.$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} f(n)$ and $\sum_{n=2}^{\infty} a^n f(a^n)$ converge or diverge together (where $a > 1$).

$$\therefore \sum_{n=2}^{\infty} a^n f(a^n) = \sum_{n=2}^{\infty} a^n \frac{1}{a^n \sqrt{\log a^n}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n \cdot \log a}}$$

$$= \frac{1}{\sqrt{\log a}} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

$\because$ $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent $\Rightarrow \sum_{n=2}^{\infty} a^n f(a^n)$ is divergent

$\Rightarrow \sum_{n=2}^{\infty} f(n)$ is divergent. #

Thank you